Choquard Equations with Mixed Potential

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Abstract

In this paper, we study the following class of nonlinear Choquard equation,

$$-\Delta u + a(z)u = K(u)f(u) \quad \text{in} \quad \mathbb{R}^N,$$

where $\mathbb{R}^N = \mathbb{R}^L \times \mathbb{R}^M$, $L \geq 2$, $K(u) = |.|^{-\gamma} * F(u)$, $\gamma \in (0, N)$, a is a continuous real function and F is the primitive function of f. Under some suitable assumptions mixed on the potential a. We prove existence of a nontrivial solution for the above equation.

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1 Introduction and main results

The aim of this paper is to study the existence of nontrivial solutions for the following nonlinear Choquard equation family

$$(P)\begin{cases} -\Delta u + a(z)u = K(u)f(u), & \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \quad u > 0 & \text{em } \mathbb{R}^N \end{cases}$$

where $K(u) = |.|^{-\gamma} * F(u)$, $\gamma \in (0, N)$, a is a nonnegative continuous real function and F is the primitive function of f.

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This problem comes from observation of waves solutions for a nonlinear Schrodinger equation of the kind

$$i\partial_t \Psi = -\Delta \Psi + W(z)\Psi - (Q(z) * |\Psi|^q)|\Psi|^{q-2}\Psi, \quad \text{in} \quad \mathbb{R}^N. \tag{1}$$

In this context, W is the external potential and Q is the response function possesses information on the mutual interaction between the bosons. This type of nonlocal equation is known to influence the propagation of electromagnetic waves in plasmas [8] and also plays an important role in the theory of Bose-Einstein condensation [10]. It is clear that $\Psi(z,t) = u(z)e^{-iEt}$ solves the evolution equation (1) if, and only if, u solves

$$-\Delta u + a(z)u = (Q(z) * |u|^q)|u|^{q-2} \quad \text{in} \quad \mathbb{R}^N, \tag{2}$$

with $a(z) = W(z) - E$.

When the response function is the Dirac function, i.e. $Q(z) = \delta(z)$, the nonlinear response is local and we have the Schrödinger equation

$$-\Delta u + a(z)u = |u|^{q-2}u \quad \text{in} \quad \mathbb{R}^N. \tag{3}$$

This equation has been studied extensively under various hypotheses on the potentials and the nonlinearities. We may refer to [4], [3], [6], [9], [7] and the references therein.

In this paper, we study the existence of nontrivial solutions for a class of Schrödinger equation with nonlocal type nonlinearities, that is, the response function Q in (P) is of Coulomb type, for example $|z|^{-\gamma}$, then we arrive at the Choquard-Pekar equation,

$$-\Delta u + a(z)u = \left(\frac{1}{|z|^\gamma} * |u|^q\right)|u|^{q-2} \quad \text{in} \quad \mathbb{R}^N. \tag{4}$$

We know that, most of the existing papers consider the existence and property of the solutions for the nonlinear Choquard equation (P) with constant potential. For example: in [13], Lieb proved the existence and uniqueness, up to translations, of the ground state solution to equation (4). Later, in [15], Lions showed the existence of a sequence of radially symmetric solutions to this equation. Involving the properties of the ground state solutions, Ma and Zhao [18] considered the generalized Choquard equation (4) for $q \geq 2$, and they proved that every positive solution of (4) is radially symmetric and monotone decreasing about some point.

Involving the problem with nonconstant potentials, we have

$$- \Delta u + a(z)u = \left(\frac{1}{|z|^{\gamma}} * F(u)\right) f(u) \quad \text{in} \quad \mathbb{R}^N, \tag{5}$$

where a is a continuous periodic function with $\inf_{\mathbb{R}^N} a(z) > 0$, noticing that the nonlocal term is invariant under translation, it is possible to prove an existence result easily by applying the Mountain Pass Theorem, see [1] for example. In [2], Alves, Figueiredo and Yang, they made a very interesting work for the case generalized Choquard equation with vanishing potential.

Looking at the various works cited above and others, we observe the lack of existence of results, for mixed potentials with different characteristics in each entry, i.e., considering $\mathbb{R}^N = \mathbb{R}^L \times \mathbb{R}^M$ and $z \in \mathbb{R}^N$ as $z = (x, y)$, $x \in \mathbb{R}^L$ and $y \in \mathbb{R}^M$, $a(x, y)$ has different characteristics for each variable. This led us to seek solution to some kinds of mixed potential.

In all cases we study, $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and satisfies:

(f_0) $|f(s)| \leq C_0(|s|^{q_1-1} + |s|^{q_2-1})$, where $q_1, q_2 > 1$ with $\frac{2N-\gamma}{N} < q_1 \leq q_2 < \frac{2N-\gamma}{N-2}$;

(f_1) $\frac{f(t)}{t}$ is increasing and unbounded in $t > 0$;

(f_2) $f(t) > 0$ in $t > 0$ and $f(t) = 0$ in $t \leq 0$.

The first mixed potential is the symmetric-coercive type, i.e., $a : \mathbb{R}^L \times \mathbb{R}^M \longrightarrow \mathbb{R}$ continuous, where $L \geq 2$, such that,

(a_0) There exists $R > 0$ such that $a(x, y) \geq a_0$, for all $z \in B_R^c(0)$, $z = (x, y)$ and $a(z) \geq 0$, for all $z \in \mathbb{R}^N$,

(a_1) $a(x, y) \longrightarrow \infty$, when $|y| \longrightarrow \infty$ uniformly for $x \in \mathbb{R}^L$;

(a_2) $a(x, y) = a(x', y)$ for all $x, x' \in \mathbb{R}^L$ with $|x| = |x'|$ and all $y \in \mathbb{R}^M$.

In this case, the obtained solution $u \in H^1(\mathbb{R}^N)$ is such that, $u(x, y) = u(x', y)$, always that $x, x' \in \mathbb{R}^L$ with $|x| = |x'|$ and all $y \in \mathbb{R}^M$. To do this, we will make use of [19], wherein the authors prove a compactness embedding lemma and a principle of symmetric criticality. In fact, we prove:

Theorem 1. *Assume $(a_0) - (a_2)$ and $(f_0) - (f_2)$. Then, problem (P) has a positive solution.*

The second mixed potential is the periodic-coercive type, i.e., $a : \mathbb{R}^L \times \mathbb{R}^M \longrightarrow \mathbb{R}$ continuous, such that, satisfies (a_0), (a_1) and

(a_3) $a(x,y) = a(x+p,y)$, for all $x \in \mathbb{R}^L, y \in \mathbb{R}^M$ and $p \in \mathbb{Z}^L$.

In fact, we prove:

Theorem 2. *Assume $(a_0),(a_1)$ and (a_3) for a and $(f_0)-(f_2)$ for f. We have that, the problem (P) has a positive solution in the level of mountain pass.*

The third and last mixed potential we work is the asymptotically periodic-coercive type, i.e., $a : \mathbb{R}^L \times \mathbb{R}^M \longrightarrow \mathbb{R}$ continuous, such that, satisfies (a_0), (a_1) and

(a_4) There exists a potential $a_p : \mathbb{R}^L \times \mathbb{R}^M \longrightarrow \mathbb{R}$ continues such that, satisfies (a_0), (a_1), (a_3) and

1. There exists $\mu > 0$ such that,

$$a(z) \lneq a_p(z) \leq \mu a(z), \quad \text{for all} \quad z \in \mathbb{R}^N$$

where, "$a(z) \lneq a_p(z)$", means that, there exists $\Omega \subset \mathbb{R}^N$ with $|\Omega| > 0$, such that, $a(z) < a_p(z)$ for all $z \in \Omega$;

2. $|a(x,y) - a_p(x,y)| \longrightarrow 0$, when $|y| \longrightarrow \infty$, uniformly for $x \in \mathbb{R}^L$.

In fact, we prove:

Theorem 3. *Assume $(a_0),(a_1)$ and (a_4) for a and $(f_0)-(f_2)$ for f. Then, problem (P) has a positive solution.*

Remark 1.1. *In our study, it was not necessary that the nonlinearity f verifies the Ambrosetti-Rabinowitz type superlinear condition for nonlocal problem, see [5], that is, there exists $\theta > 2$, such that*

$$0 < \theta F(s) \leq 2f(s)s, \quad \textit{for all} \quad s > 0,$$

*we often found this hypothesis in work on this subject. Under this condition, all results **theorem 1**, **theorem 2** and **theorem 3** are true replacing (f_1) by Ambrosetti-Rabinowitz condition.*

Remark 1.2. *A common hypothesis in the work when do not have the Ambrosetti-Rabinowitz condition, is the condition*

$$\lim_{t \longrightarrow \infty} \frac{F(t)}{t^2} = \infty,$$

but, it is easy to see that (f_1) implies the above condition.

Notations

We fix the following notations, which will use from now on.

- $z \in \mathbb{R}^N = \mathbb{R}^L \times \mathbb{R}^M$ is given by $z = (x, y)$, such that, $x \in \mathbb{R}^L$ and $y \in \mathbb{R}^M$.
- $B_R(z)$ denotes the ball centered at the z with radius $R > 0$ in $\mathbb{R}^N$.
- $L^s(\mathbb{R}^N)$, for $1 \leq s \leq \infty$, denotes the Lebesgue space with the norms

$$|u|_s = \left(\int_{\mathbb{R}^N} |u|^s dz\right)^{\frac{1}{s}}$$

or

$$|u|_s = \left(\int_{\mathbb{R}^L \times \mathbb{R}^M} |u|^s dxdy\right)^{\frac{1}{s}}.$$

- $C_0^\infty(\mathbb{R}^N)$ denotes the space of the functions infinitely differentiable with compact support in $\mathbb{R}^N$.
- We denote the inner product of $H^1(\mathbb{R}^N)$ by

$$(u, v)_{H^1} = \int_{\mathbb{R}^N} \nabla u \nabla v + uvdz$$

and the norm

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 dz\right)^{\frac{1}{2}}.$$

- From the assumptions on a, it follows that the subspace

$$E_a = \{u \in H^1(\mathbb{R}^N); \quad \int_{\mathbb{R}^N} a(z)|u|^2 dz < \infty\}$$

is a Hilbert space with norm defined by

$$\|u\| := \left(\int_{\mathbb{R}^N} |\nabla u|^2 + a(z)|u|^2 dz\right)^{\frac{1}{2}}$$

and $E_a \hookrightarrow H^1(\mathbb{R}^N)$, continuously.

2 Common properties of the problem with Mixed Potential

We would like to make some comments on the assumptions involving the nonlinearity f. We intend to use variational methods, this way, we must have:

$$\left|\int_{\mathbb{R}^N} (|z|^{-\gamma} * F(u))F(u)dz\right| < \infty, \quad \text{for all} \quad u \in E_a.$$

To see that above property occurs, it is very important to recall the Hardy-Littlewood-Sobolev inequality, found in [14], which will be frequently used in the paper.

Theorem 4. *Let $p, r > 0$ and $\gamma \in (0, N)$ with $1/p + \gamma/N + 1/r = 2$. If $f \in L^p(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$, then there exists a sharp constant $C := C(p, N, \gamma, r) > 0$, independent of f and h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)h(y)}{|x-y|^\gamma}dydx \leq C|f|_p|h|_r.$$

By (f_0),

$$|F(u)| \leq C_0(|u|^{q_1} + |u|^{q_2}) \Longrightarrow |F(u)|^{\frac{2N}{2N-\gamma}} \leq C_0(|u|^{q_1\frac{2N}{2N-\gamma}} + |u|^{q_2\frac{2N}{2N-\gamma}})$$

as,

$$2 < q_1\frac{2N}{2N-\gamma} \leq q_2\frac{2N}{2N-\gamma} < 2^*$$

thence, $F(u) \in L^{\frac{2N}{2N-\gamma}}(\mathbb{R}^N)$. Now, note that,

$$\frac{2N-\gamma}{N} + \frac{\gamma}{N} = 2,$$

thereby, by Hardy-Littlewood-Sobolev inequality,

$$\left|\int_{\mathbb{R}^N} K(u)F(u)dz\right| \leq C_0|F(u)|^2_{\frac{2N}{2N-\gamma}} < \infty.$$

From the above commentaries, the Euler-Lagrange functional $I : E_a \longrightarrow \mathbb{R}$ associated to (P) given by

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2 + a(z)|u|^2dz - \frac{1}{2}\int_{\mathbb{R}^n}K(u)F(u)dz,$$

is well defined and belongs to C^1 with its derivative given by

$$I'(u)\varphi = \int_{\mathbb{R}^N}\nabla u\nabla\varphi + a(z)u\varphi dz - \int_{\mathbb{R}^N}K(u)f(u)\varphi dz,$$

for all $u, \varphi \in E_a$. Thus, it is easy to see that all the solutions of (P) correspond to critical points of the energy functional I.

We have that, I verifies the mountain pass geometry, through of arguments well know in the literature.

Lemma 5. *Assume $\gamma \in (0, N)$, (a_0) and $(f_0) - (f_2)$. Then,*

(1) *There exist $\rho, \delta_0 > 0$ such that $I|_{S_\rho} \geq \delta_0 > 0$;*

(2) *There exist $\varphi \in E_a$ with $\|\varphi\| > \rho$ such that $I(\varphi) < 0$.*

By the mountain pass theorem, see [11], there is a Cerami sequence $(u_n) \subset E_a$, such that

$$I(u_n) \longrightarrow c_a \quad \text{e} \quad I'(u_n)(u_n) \longrightarrow 0,$$

where

$$c_a := \inf_{\alpha\in\Gamma}\max_{t\in[0,1]} I(\alpha(t)) > 0$$

and

$$\Gamma := \{\alpha \in C([0,1], E_a); \quad \alpha(0) = 0 \quad \text{e} \quad I(\alpha(1)) < 0\}.$$

The following lemma will be useful to prove our results.

Lemma 6. *Let $(u_n) \subset E_a$ such that $(I(u_n))$ is bounded and $\|u_n\| \longrightarrow \infty$, then, $w_n = \frac{u_n}{\|u_n\|}$ is such that $w_n \rightharpoonup w$ in E_a, where $w \le 0$ a.e. in $\mathbb{R}^N$.*

Proof. Since (w_n) is bounded in E, there are $w \in E_a$ and a subsequence, still denote by (w_n), such that $(w_n) \rightharpoonup w$ in E_a. For all $R > 0$, we define

$$\Omega := \{z \in \mathbb{R}^N; \quad |z| \le R \quad \text{e} \quad w(z) > 0\}.$$

We claim that $|\Omega| = 0$, for all $R > 0$. Suppose by contradiction that $|\Omega| > 0$, for some $R > 0$. Thus,

$$I(u_n) = c + o_n(1) \Longrightarrow \int_{\mathbb{R}^N} K(u_n)\frac{F(u_n)}{\|u_n\|^2}dz = 1 + o_n(1)$$

from where it follows that,

$$\int_{\Omega} K(u_n)\frac{F(u_n)}{\|u_n\|^2}dz \le 1 + o_n(1).$$

By **Remark 1.2**, for all $M > 0$, there exists $\delta > 0$ such that

$$\frac{F(s)}{s^2} \ge M, \quad \text{for all} \quad s \ge \delta$$

thence, for

$$G_n := \{z \in \mathbb{R}^N; \quad u_n(z) \ge \delta\}$$

we have to,

$$\int_{\Omega\cap G_n} K(u_n)\frac{F(u_n)}{u_n^2}w_n^2 dz \le 1 + o_n(1)$$

where,

$$M\int_{\Omega\cap G_n} K(u_n)w_n^2 dz \le 1 + o_n(1).$$

Note that, for $z \in \Omega$, from n large enough $u_n(z) \ge \delta$, because, $w_n(z) \longrightarrow w(z)$ and $u_n(z) = \|u_n\|w_n(z)$. Now, for $z \in \Omega$, n large enough,

$$K(u_n)(z) \ge F(\delta)\int_{\mathbb{R}^N} \chi_{\Omega\cap G_n}(z')|z - z'|^{-\gamma}dz'$$

and so

$$\liminf K(u_n)(z) \ge F(\delta) > 0.$$

Therefore,

$$M \liminf \int_{\Omega \cap G_n} K(u_n) w_n^2 dz \leq 1$$

by Fatou lemma,

$$MF(\delta) \int_{\Omega} w^2 dz \leq 1, \quad \text{for all} \quad M > 0,$$

consequently, $|\Omega| = 0$. ■

3 Symmetric-Coercive Case

In this case, due to the lack of compactness, we restrict I to a subspace of E_a, given by

$$E =: \{u \in E_a; \quad u(x, y) = u(x', y), |x| = |x'|\}$$

thus, E is Hilbert space under the scalar product,

$$(u, v)_E := \int_{\mathbb{R}^N} (\nabla u \nabla v + a(z) uv) dz.$$

Thus, we have the lemma:

Lemma 7. *E is continuously immersed in $L^s(\mathbb{R}^N)$ if $s \in [2, 2^*]$ and compactly immersed if $s \in (2, 2^*)$.*

Indeed, first of all, we are going to prove that if condition (a_0) is valid then the Banach space E is continuously immersed in $L^s(\mathbb{R}^N)$ for all $s \in [2, 2^*]$. Notice that (a_0) yields that,

$$\int_{\mathbb{R}^N} |u|^2 dz = \int_{|z| \leq R} |u|^2 dz + \int_{|z| > R} |u|^2 dz \leq \int_{|z| \leq R} |u|^2 dz + a_0^{-1} \int_{|z| > R} a(z) |u|^2 dz \leq C \|u\|_E^2,$$

where we use the continuity of the Sobolev embedding for bounded domains. On the other hand, the Sobolev-Gagliardo-Nirenberg inequality asserts that there exists positive constant S such that

$$\int_{\mathbb{R}^N} |u|^{2^*} dz \leq S \int_{\mathbb{R}^N} |\nabla u|^2 dz.$$

Therefore, from inequalities above, we have the continuity of the embedding for $s = 2$ and $s = 2^*$. The continuity of the immersion for a fixed $s \in (2, 2^*)$, follows from the following interpolation inequality

$$|u|_s \leq |u|_2^{1-t} |u|_{2^*}^t$$

where $t \in [0, 1]$ is such that $1/s = (1-t)/2 + t/2^*$.

Now, assuming $(a_0) - (a_2)$, we are going to prove the compactness of the embedding of the spaces E in $L^s(\mathbb{R}^N)$ for $s \in (2, 2^*)$. Let $(u_n) \subset E$ bounded, so, (u_n) is bounded in $H^1(\mathbb{R}^N)$, thus, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Setting $v_n = u_n - u$, we have, by Lion's Lemma:

(A) There exist $(z_n) \subset \mathbb{R}^N$ and $\rho, R > 0$, such that

$$\int_{B_R(z_n)} v_n^2 dz \geq \rho > 0 \quad \text{for all} \quad n \in \mathbb{N},$$

or

(B) $v_n \longrightarrow 0$ in $L^q(\mathbb{R}^N)$, for all $q \in (2, 2^*)$.

If (A) occurs, we have that (y_n) is bounded in $\mathbb{R}^M$. Indeed (v_n) is bounded, thus, there exists $\Lambda > 0$ such that,

$$\frac{\|v_n\|^2}{\Lambda} < \rho, \quad \text{for all} \quad n \in \mathbb{N}$$

so, noticing that, there exists $\lambda > 0$ such that,

$$a(x, y) \geq \Lambda, \quad \text{for all} \quad y \in B^c_\lambda(0)$$

*if (y_n) is **unbounded**, for n large enough, then*

$$(x, y) \in B_R(x_n, y_n) \Longrightarrow |y| \geq \lambda \Longrightarrow a(x, y) \geq \Lambda$$

thus,

$$0 < \rho \leq \int_{B_R(x_n, y_n)} |v_n|^2 dz \leq \frac{1}{\Lambda} \int_{B_R(x_n, y_n)} a(z)|v_n|^2 dz \leq \frac{\|v_n\|^2}{\Lambda} < \rho,$$

which is a contradiction, therefore, (y_n) is bounded. And so, there exists $\lambda > 0$ such that $|y_n| \leq \lambda$ for all $n \in \mathbb{N}$. Therefore, for $\overline{R} = R + \lambda$, we have

$$B_R(x_n, y_n) \subset B_{\overline{R}}(x_n, 0) \Rightarrow \int_{B_{\overline{R}}(x_n, 0)} |v_n|^2 dz \geq \rho > 0, \quad \text{for all} \quad n \in \mathbb{N}.$$

If (x_n) is unbounded, without loss of generality we can assume that, for every $n \in \mathbb{N}$, $|x_n| \geq n\overline{R}$. Thus, there are at least $3n$ balls of ray $\overline{R}$, disjointed and centered at points $(\overline{x}, 0)$, for $|\overline{x}| = |x_n|$. We denote by J such finite set of centers, and so

$$\int_{B_{\overline{R}}(x_n, 0)} |v_n|^2 dz = \int_{B_{\overline{R}}(\overline{x}, 0)} |v_n|^2 dz, \quad \text{for all} \quad (\overline{x}, 0) \in J$$

thus,

$$\int_{\mathbb{R}^N} |v_n|^2 dz \geq \sum_{(\overline{x}, 0) \in J} \int_{B_{\overline{R}}(\overline{x}, 0)} |v_n|^2 dz \geq 3n\rho$$

and this is a contradiction because, (v_n) is bounded. Therefore, (x_n) is bounded. Which also creates an absurd. And so, (B) is valid and we have the compact immersion.

The proof of these immersions can be made using **Lemma III.2** pp. 321 of [17], but, we choose this, because it is simpler.

We seek critical point of $I|_E$, and by principle of symmetric criticality in [19], this point is critical in $I : E_a \longrightarrow \mathbb{R}$.

It is very important to note that, the lemmas above are valid, replacing E_a by E. Now,

Lemma 8. *If $v_n \rightharpoonup v$ in E, then*

$$\lim_{n\longrightarrow\infty}\int_{\mathbb{R}^N} K(v_n)F(v_n)dz = \int_{\mathbb{R}^N} K(v)F(v)dz$$

and

$$\lim_{n\longrightarrow\infty}\int_{\mathbb{R}^N} K(v_n)f(v_n)v_n dz = \int_{\mathbb{R}^N} K(v)f(v)vdz$$

Proof. Note that, $v_n \rightharpoonup v$ in E. Thus, from compact immersion of E in $L^s(\mathbb{R}^N)$ for $s \in (2, 2^*)$, $v_n(z) \longrightarrow v(z)$ and $F(v_n(z)) \longrightarrow F(v(z))$, *a.e.* in $\mathbb{R}^N$ and $v_n \longrightarrow v$ in $L^t(\mathbb{R}^N)$, $t \in (2, 2^*)$. Now, recalling that,

$$|F(v_n)|^{\frac{2N}{2N-\gamma}} \leq C_0(|v_n|^{q_1\frac{2N}{2N-\gamma}} + |v_n|^{q_2\frac{2N}{2N-\gamma}})$$

and

$$2 < q_1\frac{2N}{2N-\gamma} \leq q_2\frac{2N}{2N-\gamma} < 2^*,$$

we have, using the dominated convergence theorem and the fact that $\Sigma : L^{\frac{2N}{2N-\gamma}}(\mathbb{R}^N) \longrightarrow L^{\frac{2N}{\gamma}}(\mathbb{R}^N)$, given by $\Sigma(w) := |.|^{-\gamma} * w$ is a linear bounded operator, that

$$K(v_n) \longrightarrow K(v), \quad \text{em} \quad L^{\frac{2N}{\gamma}}(\mathbb{R}^N).$$

Using similar arguments, we show that

$$f(v_n)v_n \longrightarrow f(v)v \quad \text{and} \quad f(v_n)v \rightharpoonup f(v)v \quad \text{in} \quad L^{\frac{2N}{2N-\gamma}}(\mathbb{R}^N).$$

Therefore,

$$\int_{\mathbb{R}^N} K(v_n)f(v_n)v_n dz \longrightarrow \int_{\mathbb{R}^N} K(v)f(v)vdz$$

and

$$\int_{\mathbb{R}^N} K(v_n)f(v_n)vdz \longrightarrow \int_{\mathbb{R}^N} K(v)f(v)vdz.$$

■

Lemma 9. *Let $(u_n) \subset E$ such that $I(u_n) \longrightarrow c_a$ and $I'(u_n)(u_n) \longrightarrow 0$, then, (u_n) is bounded in E.*

Proof. Indeed, otherwise, $\|u_n\| \longrightarrow \infty$. Setting, $w_n = u_n/\|u_n\|$, we have from **Lemma 6**

$$w_n \rightharpoonup w \quad \text{in} \quad E,$$

with $w \leq 0$. We can check that, **Lemma 8** implies that

$$\lim_{n \longrightarrow \infty} \int_{\mathbb{R}^N} K(w_n) f(w_n) w_n dz = 0.$$

Now, note that, for all $n \in \mathbb{N}$,

$$\text{there exists} \quad t_n \in [0,1]; \quad I(t_n u_n) = \max_{s \in [0,1]} I(s u_n).$$

Thus, given $R > 0$, for n large enough,

$$I(t_n u_n) \geq I\left(\frac{R}{\|u_n\|} u_n\right) = \frac{R^2}{2} - \frac{1}{2} \int_{\mathbb{R}^N} K(Rw_n) F(Rw_n) dz$$

so,

$$I(t_n u_n) \geq \frac{R^2}{2} + o_n(1)$$

from where it follows that,

$$\liminf_{n \longrightarrow \infty} I(t_n u_n) = \infty,$$

as $I(0) = 0$ and $I(u_n) \longrightarrow c_a$, we have that $t_n \in (0,1)$. And so, $I'(t_n u_n)(u_n) = 0$ and $I'(t_n u_n)(t_n u_n) = 0$. Thus,

$$4I(t_n u_n) = 4I(t_n u_n) - I'(t_n u_n)(t_n u_n) = t_n^2 \|u_n\|^2 + \int_{\mathbb{R}^N} K(t_n u_n)[f(t_n u_n)(t_n u_n) - 2F(t_n u_n)] dz$$

consequently,

$$4I(t_n u_n) \leq \|u_n\|^2 + \int_{\mathbb{R}^N} K(u_n)[f(u_n)(u_n) - 2F(u_n)] dz = 4I(u_n) - I'(u_n)(u_n) = 4I(u_n) + o_n(1),$$

a contradiction. Therefore, (u_n) is bounded.

■

Note that, we have actually proved that, the Cerami sequence (u_n) is bounded in E, and so, $u_n \rightharpoonup u$ in E. From where it follows that,

$$I'(u_n)(u_n) = o_n(1) \quad \text{and} \quad I'(u_n)(u) = o_n(1)$$

so,

$$\|u_n\|^2 = \int_{\mathbb{R}^N} K(u_n) f(u_n) u_n dz + o_n(1)$$

and

$$\|u\|^2 = \int_{\mathbb{R}^N} K(u_n) f(u_n) u dz + o_n(1).$$

Therefore,

$$\|u_n\| \longrightarrow \|u\|,$$

thence,

$$u_n \longrightarrow u \quad \text{em} \quad E.$$

Applying arguments above, easily, we have the **Theorem 1**.

4 Periodic-Coercive Case

Everything that was studied until the **Lemma 6** is hold for this new potential. Now, we prove a version of **Lemma 8** for this new potential.

Lemma 10. *Let $(u_n) \subset E_a$ such that $I(u_n) \longrightarrow c_a$ and $I'(u_n)(u_n) \longrightarrow 0$, then, (u_n) is bounded in E_a.*

Proof. Firstly, note that, without loss of generality, $u_n \geq 0$ in $\mathbb{R}^N$, for all $n \in \mathbb{N}$. We claim that, (u_n) is bounded in E_a. Indeed, otherwise, $\|u_n\| \longrightarrow \infty$. Setting, $w_n = u_n/\|u_n\|$, we have

$$w_n \rightharpoonup w \quad \text{in} \quad E_a,$$

with $w \leq 0$. By **Lion's Lemma**, one of two situations occurs:

(i) For all $q \in (2, 2^*)$,

$$\lim_{n \longrightarrow \infty} \int_{\mathbb{R}^N} w_n^q dz = 0.$$

(ii) There exists $R, \eta > 0$ and $(x_n) \subset \mathbb{Z}^L$ such that,

$$\liminf_{n \longrightarrow \infty} \int_{B_R(x_n,0)} w_n^2 dz \geq \eta > 0.$$

In truth, the Lion's Lemma, ensures that (i) or that for every $R > 0$, there exists $\delta > 0$ and $(z_n) \subset \mathbb{R}^N$, $z_n = (x_n, y_n)$, such that

$$\int_{B_R(x_n,y_n)} |w_n|^2 dz \geq \delta > 0, \quad \text{for all} \quad n \in \mathbb{N}.$$

But, note that,

Claim 4.1. *(y_n) is bounded in $\mathbb{R}^M$.*

The proof of such a claim is similar to what we did at the beginning of **section 3**.

By the claim above, there exists $\lambda > 0$ such that, $|y_n| \leq \lambda$, for all $n \in \mathbb{N}$. Then, clearly,

$$B_R(x_n, y_n) \subset B_{\tilde{R}}(x_n, 0), \quad \text{for all} \quad n \in \mathbb{N}$$

where $\tilde{R} = R + \lambda$. Thus,

$$0 < \delta \leq \int_{B_R(x_n,y_n)} |w_n|^2 dz \leq \int_{B_{\tilde{R}}(x_n,0)} |w_n|^2 dz,$$

to facilitate, replace $\tilde{R}$ by R. Now, for all $n \in \mathbb{N}$, consider $\overline{x}_n \in \mathbb{Z}^L$ such that, $|\overline{x}_n - x_n| \leq \sqrt{L}$. Then, for $\overline{R} = R + \sqrt{L}$, we have

$$B_R(x_n, 0) \subset B_{\overline{R}}(\overline{x}_n, 0),$$

hence,

$$0 < \delta \leq \int_{B_R(x_n,y_n)} |w_n|^2 dz \leq \int_{B_{\overline{R}}(\overline{x}_n,0)} |u_n|^2 dz,$$

It is then guaranteed that, indeed, (i) or (ii) is hold. If (i) occurs, the result follows as the **Lemma 5**. Now, if (ii) occurs, setting $\tilde{w}_n(z) = w_n(z+z_n)$, where $z_n = (x_n, 0)$, next, $\|\tilde{w}_n\| = \|w_n\|$, so, $(\tilde{w}_n)$ is bounded in E_a. Thus, $\tilde{w}_n \rightharpoonup \tilde{w}$ in E_a and then,

$$\liminf_{n\longrightarrow\infty} \int_{B_R(0)} \tilde{w}_n^2 dz = \liminf_{n\longrightarrow\infty} \int_{B_R(z_n)} w_n^2 dz \geq \eta > 0$$

and so,

$$\int_{B_R(0)} \tilde{w}^2 dz \geq \eta > 0$$

and $\tilde{w} \neq 0$. Now, as

$$I(u_n) = c_a + o_n(1) \Longrightarrow \int_{\mathbb{R}^N} K(u_n) \frac{F(u_n)}{\|u_n\|^2} dz = 1 + o_n(1)$$

then,

$$\int_{\mathbb{R}^N} K(u_n) \frac{F(u_n)}{u_n^2} w_n^2 dz \leq 1 + o_n(1) \Longrightarrow \int_{B_R(z_n)} K(u_n) \frac{F(u_n)}{u_n^2} w_n^2 dz \leq 1 + o_n(1).$$

By changing variables and assumption of f, defining

$$G_n = \{z \in \mathbb{R}^N; \quad u_n(z+z_n) \geq \delta\},$$

we have for $M > 0$, that

$$M \int_{B_R(0)\cap G_n} K(u_n)(z+z_n) \tilde{w}_n^2 dz \leq 1 + o_n(1).$$

On the other hand, by $F(s)/s^2$ is increasing for $s > 0$, we have

$$K(u_n)(z+z_n) \geq \|u_n\|^2 \int_{\mathbb{R}^N} \frac{F(w_n(\overline{z}))}{|\overline{z} - (z+z_n)|^\gamma} d\overline{z}$$

thus,

$$K(u_n)(z+z_n) \geq \|u_n\|^2 K(w_n)(z+z_n).$$

Therefore, by changing variables, $K(w_n)(z+z_n) = K(\tilde{w}_n)(z)$, thus,

$$1 + o_n(1) \geq M\|u_n\|^2 \int_{B_R(0)\cap G_n} K(\tilde{w}_n)(z) \tilde{w}_n^2 dz = M\|u_n\|^2 \int_{B_R(0)} \chi_{G_n}(z) K(\tilde{w}_n)(z) \tilde{w}_n^2 dz.$$

By the properties of F,

$$K(\tilde{w}_n)(z) \geq \frac{F(\delta)}{\|u_n\|^2} \int_{\mathbb{R}^N} \chi_{B_R(0)\cap G_n}(\overline{z}) |\overline{z} - z|^{-\gamma} d\overline{z},$$

thus,

$$1 + o_n(1) \geq M.F(\delta) \int_{B_R(0)} \chi_{G_n}(z) \tilde{w}_n^2(z) X_n(z) dz$$

where,

$$X_n(z) = \int_{\mathbb{R}^N} \chi_{B_R(0)\cap G_n}(\overline{z})|\overline{z} - z|^{-\gamma} d\overline{z}.$$

For $z \in B_R(0)$, a. e., $z \in G_n$ for n large and $\liminf_{n\longrightarrow\infty} X_n(z) > 0$, we got that,

$$1 \geq M_\delta \int_{B_R(0)} \tilde{w}^2 dz \geq M_\delta \eta > 0.$$

A contradiction. Therefore, (u_n) is bounded.

■

We got that, $(u_n) \subset E_a$ is bounded, so, $u_n \rightharpoonup u$ in E_a. Clearly, $I'(u) = 0$, then, u is solution of problem, not necessarily nontrivial. Thus, we need,

Lemma 11. *There exist $(x_n) \subset \mathbb{Z}^L$, such that $(w_n) \subset E_a$, given by $w_n(x, y) = u_n(x + x_n, y)$, weakly converges to a function $w \in E_a$ with $w \neq 0$ and $I'(w) = 0$. In other words, w is a solution nontrivial.*

Proof. Firstly, we have the claim:

Claim 4.2. *For all $R > 0$, there exists $\delta > 0$ and $(z_n) \subset \mathbb{R}^N$ such that*

$$\int_{B_R(z_n)} |u_n|^2 dz \geq \delta > 0, \quad \textit{for all} \quad n \in \mathbb{N}.$$

Indeed, otherwise, by **Lion's Lemma**, $u_n \longrightarrow 0$ in $L^q(\mathbb{R}^N)$, for all $q \in (2, 2^*)$. Thus, $u_n \longrightarrow 0$ in E_a. A contradiction. Consequently, similarly to what we did in **Lemma 10**, there exist $R, \delta > 0$ and $(x_n) \subset \mathbb{Z}^L$ such that,

$$\int_{B_R(x_n,0)} u_n^2 dz \geq \delta > 0.$$

Setting, $w_n(x, y) = u_n(x + x_n, y)$, we have $\|w_n\| = \|u_n\|$, and so, as (u_n) is bounded, (w_n) is bounded too. Hence, $w_n \rightharpoonup w$ in E_a, consequently, $w_n \longrightarrow w$ in $L^s_{loc}(\mathbb{R}^N)$, for all $s \in [2, 2^*]$. Thus, $w_n \longrightarrow w$ in $L^2(B_R(0))$, but,

$$0 < \delta \leq \int_{B_R(x_n,0)} |u_n|^2 dz = \int_{B_R(0)} |w_n|^2 dz$$

and so, $w \neq 0$.

Claim 4.3. *w is critical point of I.*

Indeed, note that

$$I(w_n) = \frac{1}{2} A_n - \frac{1}{2} B_n$$

where,

$$A_n = \int_{\mathbb{R}^N} |\nabla w_n|^2 + a(z)|w_n|^2 dz \quad \text{and} \quad B_n = \int_{\mathbb{R}^N} K(w_n) F(w_n) dz.$$

By periodicity,

$$A_n = \int_{\mathbb{R}^N} |\nabla u_n|^2 + a(z)|u_n|^2 dz$$

and, by changing variables, we have

$$B_n = \int_{\mathbb{R}^N} K(u_n)(z)F(u_n)(z)dz.$$

Hence,

$$I(w_n) = I(u_n) \longrightarrow c_a.$$

On the other hand, for $\varphi \in E_a$ with $\|\varphi\| \leq 1$, proceeding analogously, we have

$$|I'(w_n)\varphi| = |I'(u_n)\varphi(z - z_n)| \leq \|I'(u_n)\|.\|\varphi\| \leq \|I'(u_n)\|$$

where consider $z_n = (x_n, 0)$ and , then, $\|I'(w_n)\| \longrightarrow 0$ when $n \longrightarrow \infty$. Therefore, w is a nontrivial critical point of problem.

∎

By assumptions on f, we have $w \geq 0$, thus, by the weak maximum principle, in [12], we have $w > 0$.

We find the solution we wanted, but, we are not sure of being a ground state solution. But, note that, $w_n \rightharpoonup w$ in E_a, and so

$$w_n(z) \longrightarrow w(z) \quad \text{and} \quad K(w_n)f(w_n)w_n(z) \longrightarrow K(w)f(w)w(z) \quad \text{a.e. in } \mathbb{R}^N.$$

Moreover,

$$\|w\|^2 = \int_{\mathbb{R}^N} K(w)f(w)wdz,$$

thus, by Fatou's Lemma,

$$I(w) = I(w) - \frac{1}{2}I'(w)(w) \leq \varliminf_{n \longrightarrow \infty} \left(I(w_n) - \frac{1}{2}I'(w_n)w_n \right) = c_a,$$

therefore, $I(w) \leq c_a$.

To be w ground state solution, we have the theorem below:

Theorem 12. *Let $N = \{u \in E_a; \quad I'(u)(u) = 0\} \setminus \{0\}$, the Nehari manifold of I. Then,*

$$c_a \leq \inf_N I.$$

Proof. Let $u \in N$. For $t \geq 0$, we got

$$I(tu) = \frac{t^2}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} K(tu)F(tu)dz$$

and

$$0 = I'(u)(u) = \|u\|^2 - \int_{\mathbb{R}^N} K(u)f(u)udz.$$

Moreover, using properties of f, we have that

$$\frac{d}{dt}[I(tu)] > 0, \quad \text{if } t < 1 \text{ and } \quad \frac{d}{dt}[I(tu)] < 0, \quad \text{if } t > 1.$$

Therefore,

$$\max_{t \geq 0} I(tu) = I(u)$$

and so,

$$\inf_{\alpha \in \Gamma} \max_{t \in [0,1]} I(\alpha(t)) \leq I(u), \quad \forall u \in N.$$

Consequently,

$$c_a \leq \inf_N I.$$

■

This proves the existence of a solution in the mountain pass level, i.e., thus proving the **theorem 2**.

5 Asymptotically Periodic-Coercive Case

Note that, associated with potential a_p, we have the problem:

$$(Q) \begin{cases} -\Delta u + a_p(z)u = K(u)f(u), \quad \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N), \quad u > 0 \quad \text{in } \mathbb{R}^N \end{cases}$$

As in the problem (P), we must find solution in a space of type E_a, i.e., in the E_{a_p}, analogous to E_a. Now, note that, $E_a = E_{a_p}$ and, $\|.\|_{E_a}$ and $|.\|_{E_{a_p}}$ are equivalents.

Consequently, we have that, the energy functional of problems (P) and (Q) are $I, J : E \longrightarrow \mathbb{R}$, defined respectively by:

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + a(z)|u|^2 dz - \frac{1}{2}\int_{\mathbb{R}^N} K(u)F(u)dz$$

and

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + a_p(z)|u|^2 dz - \frac{1}{2}\int_{\mathbb{R}^N} K(u)F(u)dz.$$

These energy functionals check up the mountain pass geometry. For I, exist $(u_n) \subset E$ such that,

$$I(u_n) \longrightarrow c_a \quad \text{and} \quad I'(u_n)(u_n) \longrightarrow 0.$$

Similarly to what we have already done, (u_n) is bounded in E, with any of the norms $\|.\|_{E_a}$ and $|.\|_{E_{a_p}}$. And so, $u_n \rightharpoonup u$ in E, $u_n(z) \longrightarrow u(z)$ a.e. in $\mathbb{R}^N$ and $u_n \longrightarrow u$ in $L^s_{loc}(\mathbb{R}^N)$, for $s \in [1, 2^*)$. Moreover, $I'(u) = 0$. We need to ensure that $u \neq 0$.

Claim 5.1. $u \neq 0$.

By the previous case, there exists $w \in E$ positive, such that

$$J(w) = c_{a_p} \quad \text{and} \quad J'(w) = 0$$

where c_{a_p} is the level of Mountain Pass for J.

Lemma 13. *The levels c_a and c_{a_p} satisfies $c_a < c_{a_p}$.*

Proof. Clearly, by definition, we have that $c_a \leq c_{a_p}$. For w, consider the path

$$\delta_w(t) = tsw, \quad \text{for all} \quad t \in [0,1] \quad (\delta_w \in \Gamma)$$

where s is fixed, such that

$$I(sw) < 0.$$

So,

$$c_a \leq \max_{t\in[0,1]} I(\delta_w(t)) = \max_{t\in[0,1]} I(tsw) \leq \max_{r\geq 0} I(rw) = I(r_0 w) < J(r_0 w) \leq \max_{r\geq 0} J(rw) = J(w),$$

thus,

$$c_a < J(w) = c_{a_p}.$$

■

Now, note that, if $u = 0$, we have:

Lemma 14.

$$|I(u_n) - J(u_n)| \longrightarrow 0 \quad \textit{and} \quad \|I'(u_n) - J'(u_n)\| \longrightarrow 0.$$

It is very important to highlight, as (u_n) is bounded in E, that there exists $M > 0$ such that $\|u_n\| \leq M$, for all $n \in \mathbb{N}$. Thus, as for all $\varepsilon > 0$, there exists $R > 0$, such that

$$a(x,y) \geq \frac{1}{\varepsilon}, \quad |y| \geq R,$$

we have,

$$\int_{|y|\geq R} |u_n|^2 dz \leq \int_{|y|\geq R} \varepsilon a(z)|u_n|^2 dz \leq \varepsilon \int_{|y|\geq R} a(z)|u_n|^2 dz \leq \varepsilon M,$$

for interpolation,

$$\int_{|y|\geq R} |u_n|^q dz < C\varepsilon, \quad \text{for all} \quad q \in [2, 2*)$$

and consequently,

$$\left| \int_{|y|\geq R} K(u_n) f(u_n) u_n dz \right| < C\varepsilon$$

where the above properties are valid for all $n \in \mathbb{N}$, and the constant $C > 0$.

For $R > 0$ large enough, define $\varphi_R \in C^\infty(\mathbb{R}^M)$, such that $|\nabla \varphi_R| \leq 2/R$, $\varphi_R(y) = 0$ if $|y| \leq R/2$, $\varphi_R(y) = 1$ if $|y| \geq R$ and $0 \leq \varphi_R(y) \leq 1$, for all $y \in \mathbb{R}^M$. We define also, $v_n(x, y) = u_n(x, y)\varphi_R(y)$, so, $\nabla v_n = (\nabla u_n)\varphi_R + u_n(\nabla \varphi_R)$, clearly, (v_n) is bounded in E and, consequently, we have that

$$o_n(1) = I'(u_n)(v_n) = \int_{\mathbb{R}^N} |\nabla u_n|^2 \varphi_R + a(z)u_n^2 \varphi_R dz + \int_{\mathbb{R}^N} u_n \nabla \varphi_R \nabla u_n dz - \int_{\mathbb{R}^N} K(u_n) f(u_n) u_n \varphi_R dz.$$

Noting that, for $\varepsilon > 0$ small enough and $R > 0$ large enough, we have

$$\left| \int_{\mathbb{R}^N} u_n \nabla \varphi_R \nabla u_n dz \right| < \varepsilon$$

as well as,

$$\left| \int_{\mathbb{R}^N} K(u_n) f(u_n) u_n \varphi_R dz \right| < \varepsilon.$$

Thus, for $\varepsilon > 0$ small enough and $R > 0$ large enough

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \varphi_R + a(z)u_n^2 \varphi_R dz < \varepsilon$$

so,

$$\int_{|y|\geq R} a(z)|u_n|^2 dz < \varepsilon.$$

Similarly, you can check that

$$\int_{|y|\geq R} a_p(z)|u_n|^2 dz < \varepsilon.$$

Now, we can prove the **lemma 13**:

Proof. Indeed,

$$\begin{aligned} I(u_n) - J(u_n) &= \int_{\mathbb{R}^L \times \overline{B}_R^c(0)} [a(z) - a_p(z)]|u_n|^2 dz + \int_{\overline{B}_R(0) \times \overline{B}_R(0)} [a(z) - a_p(z)]|u_n|^2 dz \\ &\quad + \int_{\overline{B}_R^c(0) \times \overline{B}_R(0)} [a(z) - a_p(z)]|u_n|^2 dz \end{aligned}$$

thus, by asymptotic properties,

$$\left| \int_{\mathbb{R}^L \times \overline{B}_R^c(0)} [a(z) - a_p(z)]|u_n|^2 dz \right| < \frac{\varepsilon}{3}$$

as, $u_n \longrightarrow 0$ in $L^2(\overline{B}_R(0) \times \overline{B}_R(0))$ and the limitation of the potential,

$$\left| \int_{\overline{B}_R(0)\times\overline{B}_R(0)} [a(z) - a_p(z)]|u_n|^2 dz \right| < \frac{\varepsilon}{3}$$

and,

$$\left| \int_{\overline{B}^c_R(0)\times\overline{B}_R(0)} [a(z) - a_p(z)]|u_n|^2 dz \right| < \frac{\varepsilon}{3}$$

we got that,

$$|I(u_n) - J(u_n)| \longrightarrow 0.$$

Similarly,

$$\|I'(u_n) - J'(u_n)\| \longrightarrow 0.$$

■

In this way, we ensure that,

$$J(u_n) \longrightarrow c_a \quad \text{and} \quad J'(u_n)(u_n) \longrightarrow 0.$$

By **Lema 7**: exist $(\overline{x}_n) \subset \mathbb{Z}^L$, such that $(v_n) \subset E$, given by $v_n(x, y) = u_n(x + \overline{x}_n, y)$, converges weakly to $v \in E$ with $v \neq 0$ and $J'(v) = 0$. Thus, w is solution nontrivial. And so, $v \in N_J = \{u \in E; \quad J'(u)(u) = 0\} \setminus \{0\}$, in this way,

$$c_{a_p} \leq \inf_{u \in N_J} J(u) \leq J(v).$$

On the other hand,

$$\begin{aligned} J(v) &= J(v) - \frac{1}{2}J'(v)(v) \\ &\leq \varliminf_{n\longrightarrow\infty} \int_{\mathbb{R}^N} K(v_n)\frac{f(v_n)v_n}{2} - K(v_n)F(v_n)dz \\ &= \varliminf_{n\longrightarrow\infty} \left(I(u_n) - \frac{1}{2}I'(u_n)(u_n) \right) \end{aligned}$$

therefore,

$$J(v) \leq \varliminf_{n\longrightarrow\infty} I(u_n) = c_a,$$

so, $c_{a_p} \leq c_a$. A contradiction. Therefore, $u \neq 0$.

By applying the arguments above, we have **theorem 3**.

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